THE OLYMPIAD CORNER

No. 292

R.E. Woodrow

We begin the section of solutions from our readers with the file of solutions to problems of the Thai Mathematical Olympiad Examinations 2006, Selected problems, given at [2010 : 83–84].

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$f(x^2 + x + 3) + 2f(x^2 - 3x + 5) = 6x^2 - 10x + 17$$

for all real x. Find $f(85)$.

Solved by Arkady Alt, San Jose, CA, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; and Titu Zvonaru, Cománeşti, Romania. We use an edited version of the solution of Alt.

Let $p(x) = x^2 + x + 3$, $q(x) = x^2 - 3x + 5$. Since

$$p(1-x) = (1-x)^2 + (1-x) + 3 = x^2 - 3x + 5 = q(x),$$

$q(1-x) = p(1-(1-x)) = p(x)$ and

$$6(1-x)^2 - 10(1-x) + 17 = 6x^2 - 2x + 13$$

then

$$6x^2 - 2x + 13 = f(p(1-x)) + 2f(q(1-x)) = f(q(x)) + 2f(p(x))$$

and from the system of equations

$$\begin{cases} f(p(x)) + 2f(q(x)) = 6x^2 - 10x + 17 \\ 2f(p(x)) + f(q(x)) = 6x^2 - 2x + 13 \end{cases}$$

we obtain

$$\begin{aligned} 3f(p(x)) &= 2(2f(p(x)) + f(q(x))) - (f(p(x)) + 2f(q(x))) \\ &= 2\left(6x^2 - 2x + 13\right) - \left(6x^2 - 10x + 17\right) \\ &= 6x^2 + 6x + 9 \iff f(p(x)) = 2p(x) - 3. \end{aligned}$$

Since $p(x) = x^2 + x + 3 \geq \dfrac{11}{4}$ then for any $y \geq \dfrac{11}{4}$ there is x such that $p(x) = y$ and, therefore, for any $y \geq \dfrac{11}{4}$ we have

$$f(y) = f(p(x)) = 2p(x) - 3 = 2y - 3.$$

Hence, $f(85) = 167$.